Quasi-exactly solvable problems
and random matrix theory

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Abstract

There exists an exact relationship between the quasi-exactly solvable problems of quantum mechanics and models of square and rectangular random complex matrices. This relationship enables one to reduce the problem of constructing topological $(1/N)$ expansions in random matrix models to the problem of constructing semiclassical expansions for observables in quasi-exactly solvable problems. Lie algebraic aspects of this relationship are also discussed.

1 Introduction. The idea

The method of orthogonal polynomials [1] is one of the basic tools for doing calculations in the theory of large random matrices (see e.g. recent review papers [2, 3, 4]). In order to remind the reader the idea of this method, let us consider for definiteness a model of random hermitean matrices Φ_N of size $(N+1) \times (N+1)$ described by the potential $V(\Phi_N)$. The 'observables' in this model are initially defined as fractions of multiple integrals

$$\langle f \rangle = \frac{\int D\Phi_N \mathrm{Tr} \ f(\Phi_N) \exp\{-\mathrm{Tr} \ V(\Phi_N)\}}{\int D\Phi_N \exp\{-\mathrm{Tr} \ V(\Phi_N)\}} \tag{1}$$

over components of matrix Φ_N. The measure of the integration, $D\Phi_N$, as well as subintegral expressions are invariant under unitary transformations, $\Phi_N \to U_N \Phi_N U_N^{-1}$, and this fact enables one to extract from both integrals (and then cancel) the volume of the unitary group by means of the well known Faddeev – Popov method. After this, formula (1) takes simpler form

$$\langle f \rangle = \frac{\int d\lambda_0 \ldots d\lambda_N \Delta^2(\lambda) \sum_{i=0}^{N} f(\lambda_i) \exp\{-\sum_{i=0}^{N} V(\lambda_i)\}}{\int d\lambda_0 \ldots d\lambda_N \Delta^2(\lambda) \exp\{-\sum_{i=0}^{N} V(\lambda_i)\}} \tag{2}$$

in which the variables $\lambda_0, \ldots, \lambda_N$ have the meaning of eigenvalues of matrix Φ_N, and $\Delta_N(\lambda) = \det ||\lambda_i^n||$ is the Vandermonde determinant of these eigenvalues.

A brilliant idea by Bessis [1] was to represent this determinant in the form $\Delta_N(\lambda) = \det ||P_n(\lambda_i)||$, where $P_n(\lambda)$ are polynomials normalized as $P_n(\lambda) = \lambda^n + \ldots$ and orthogonal with the weight $e^{-V(\lambda)}$. This replacement enables one to reduce formula (2) to the form

$$\langle f \rangle = \sum_{n=0}^{N} f_{nn} \tag{3}$$

where

$$f_{nn} = \frac{\int dx \ f(x) \Psi_n^2(x)}{\int dx \ \Psi_n^2(x)} \tag{4}$$

and

$$\Psi_n(x) = P_n(x) e^{-V(x)/2}. \tag{5}$$

From the standard theory of orthogonal polynomials (see, e.g. [5]) it follows that, if function $f(x)$ is a polynomial of degree k, then the quantities f_{nn} can be found from simple recurrence relations of the depth $k+2$. In large N limit these recurrence relations can be reduced to a differential equation whose solution determines the leading term of the topological $(1/N)$ expansion for (1). The construction of the next terms of this topological expansion is generally rather complicated matter, because the n-dependence of numbers f_{nn} is not always smooth [6]. The idea of the approach which we intend to present in this paper is to find such random matrix models for which the functions (5) could be identified with orthogonal wavefunctions for some quantum mechanical model. In this case the quantity $\langle f \rangle$ would have the meaning of the trace of function $f(x)$ over the first N excitations in this model. In large N limit, this trace could easily be computed (with an arbitrary accuracy) in the framework of WKB approximation. In this case the computation of several first terms of the topological expansion for (1) should not encounter serious difficulties.

Of course, such an identification is not always possible, because the wavefunctions of quantum mechanical models are not obliged to have the form (5). There are however, two classes of models for which the form of wavefunctions is given by formula (5). These are *exactly solvable models* associated with classical orthogonal polynomials (see e.g. [7]) and the so-called *quasi-exactly solvable models* associated with non-standard orthogonal polynomials [8]. The corresponding random matrix models we respectively shall call *exactly-* and *quasi-exactly solvable random matrix models*.

It is not difficult to see that the exactly solvable random matrix models are really exactly solvable in the standard sense of this word. Consider, for example, the simplest case of the harmonic oscillator with the potential $W(x) = x^2$. The wavefunctions in this model have the form $\Psi_n(x) = H_n(x)e^{-x^2/2}$, where $H_n(x)$ are Hermite polynomials. Comparing this form with (5) we find the form of the random matrix potential: $V(x) = x^2$. But this potential describes the Gauss ensemble of random matrices and is exactly solvable [9]. It seems possible that other exactly solvable quantum models like Morse potential, Pöschel–Teller potential well, etc. [7], associated with other classical orthogonal polynomials, will be related to random matrix models exactly solvable for any N.

Much more non-trivial and interesting classes of random matrix models appear if one starts with the quasi-exactly solvable models in quantum mechanics [8]. Remember that these models, which have been discovered several years ago [10, 11, 12, 13, 14], are distinguished by the fact that they can be solved exactly (analytically) only for a finite number of eigenvalues and corresponding eigenfunctions. It is however remarkable that the number N of exactly calculable energy levels is a free parameter of the hamiltonian and can be chosen arbitrarily. The aim of the present paper is to describe the quasi-exactly solvable random matrix models associated with some simplest quasi-exactly solvable quantum mechanical problems.

2 Quasi-exactly solvable sextic oscillator model

Consider a one-dimensional sextic anharmonic oscillator with hamiltonian

$$H_N = -\hbar_N^2 \frac{\partial^2}{\partial x^2} + \{b^2 - [8 - 2\hbar_N]a\}x^2 + 4abx^4 + 4a^2 x^6, \tag{6}$$

in which $a > 0$ and b are real parameters and N is an arbitrarily chosen non-negative integer. The parameter $\hbar_N$ is defined as $\hbar_N \equiv (N+1)^{-1}$ and has the meaning of the "quantized" Planck constant. It is not difficult to show that the model (6) is *quasi-exactly solvable* and has the *order* $N + 1$ [8]. This means that Schrödinger equation for (6) for any given N admits exact (algebraic) solutions only for $N + 1$ energy levels and corresponding wavefunctions. An explicit form of these solutions is given by the formulas

$$E_n = (4 - 3\hbar_N)b + 8a\hbar_N \sum_{i=1}^{N} \xi_{n,i}, \qquad n = 0, \ldots, N \tag{7}$$

and

$$\Psi_n(x) = c_n \prod_{i=1}^{N} \left(x^2 - \xi_{n,i}\right) \exp\left[-\frac{ax^4 + bx^2}{2\hbar_N}\right], \qquad n = 0, \ldots, N, \tag{8}$$

in which $\{\xi_{n,1}, \ldots, \xi_{n,N}\}$, $n = 0, \ldots, N$ are $N + 1$ different sets of real numbers satisfying the system of N algebraic equations

$$\sum_{k=1, k \neq i}^{N} \frac{\hbar_N}{\xi_{n,i} - \xi_{n,k}} + \frac{\hbar_N}{4\xi_{n,i}} - \frac{b}{2} - a\xi_{n,i} = 0, \qquad i = 1, \ldots N. \tag{9}$$

It can be shown that $N + 1$ solutions of system (9) describe the first $N + 1$ even energy levels in model (6), i.e. levels with numbers $2n$, $n = 0, \ldots, 2N$. The linear span of corresponding wavefunctions forms a $N + 1$-dimensional subspace of Hilbert space which we denote by $\mathcal{H}_N$.

3 Non-standard orthogonal polynomials

We choose the normalization constants c_n in formula (8) in such a way as to guarantee the orthonormalizability of wave functions $\Psi_n(x)$:

$$\int_{-\infty}^{+\infty} \Psi_n(x)\Psi_m(x)dx = \delta_{nm}. \tag{10}$$

Representing wavefunctions in the form

$$\Psi_n(x) = \mathcal{P}_n(x^2) \exp\left\{-\frac{ax^4 + bx^2}{2\hbar_N}\right\} \tag{11}$$

where

$$\mathcal{P}_n(x^2) = c_n \prod_{i=1}^{N} \left(x^2 - \xi_{n,i}\right), \tag{12}$$

we can rewrite (10) as

$$\int_{-\infty}^{+\infty} dx \mathcal{P}_n(x^2)\mathcal{P}_m(x^2) \exp\left\{-\frac{ax^4 + bx^2}{\hbar_N}\right\} = \delta_{nm}. \tag{13}$$

From formula (13) it follows that $\mathcal{P}_n(x^2)$ can be considered as polynomials orthogonal with the weight $\exp[-(ax^4 + bx^2)/\hbar_N]$. Of course, $\mathcal{P}_n(x^2)$ are not the classical orthogonal polynomials because they are of the same degree N. The general form of these polynomials is

$$\mathcal{P}_n(x^2) = \sum_{m=0}^{N} P_{nm} x^{2m}, \qquad n = 0, \ldots, N, \tag{14}$$

where P_{nm} is a certain non-degenerate (and non-triangular!) $(N + 1) \times (N + 1)$ matrix.

It is worth stressing that in the theory of random matrices a basic tool is provided by the set of monic polynomials ($P_n = x^n$ + lower degree monomials) which are orthogonal

$$\int_{-\infty}^{+\infty} dx P_n(x^2)P_m(x^2) \exp\left\{-\frac{ax^4 + bx^2}{\hbar_N}\right\} = h_n\, \delta_{nm}. \tag{15}$$

They obey the recursion relation

$$xP_n(x) = P_{n+1} + R_n P_{n-1} \tag{16}$$

and may be considered completely known, since Bessis [1] evaluated the generating function of the momenta μ_k

$$\mu_k = \int_{-\infty}^{+\infty} dx\, x^k \exp\left\{ -\frac{ax^4 + bx^2}{\hbar_N} \right\} \tag{17}$$

and related the coefficients R_n to the momenta.

4 From quantum mechanics to random matrix theory

Let us now take an arbitrary even function $f(x^2)$ and consider its trace $\mathrm{Tr}_N f(x^2)$ in the space $\mathcal{H}_N$, i.e. in the space of all exactly calculable wavefunctions:

$$\mathrm{Tr}_N f(x^2) = \sum_{n=0}^{N} \int_{-\infty}^{+\infty} f(x^2) \Psi_n^2(x) dx. \tag{18}$$

Using (11) and (13), we can rewrite (18) as

$$\mathrm{Tr}_N f(x^2) = \sum_{n=0}^{N} \frac{\int_{-\infty}^{+\infty} dx\, f(x^2) \mathcal{P}_n^2(x^2) \exp[-\hbar_N^{-1}(ax^4 + bx^2)]}{\int_{-\infty}^{+\infty} dx\, \mathcal{P}_n^2(x^2) \exp[-\hbar_N^{-1}(ax^4 + bx^2)]}. \tag{19}$$

It is not difficult to see that this expression can also be represented in the form of a fraction of multiple integrals

$$\mathrm{Tr}_N f(x^2) = \frac{\int_{-\infty}^{+\infty} \prod_{i=1}^{N+1} dx_i \left\{ \sum_{i=1}^{N+1} f(x_i^2) \right\} \det ||\mathcal{P}_n(x_i^2)||^2 \exp\left\{ -\hbar_N^{-1} \sum_{i=1}^{N+1}(ax_i^4 + bx_i^2) \right\}}{\int_{-\infty}^{+\infty} \prod_{i=1}^{N+1} dx_i\, \det ||\mathcal{P}_n(x_i^2)||^2 \exp\left\{ -\hbar_N^{-1} \sum_{i=1}^{N+1}(ax_i^4 + bx_i^2) \right\}}. \tag{20}$$

Using formula (14), we can write

$$\det ||\mathcal{P}_n(x_i^2)|| = \det ||P_{nm} x_i^{2m}|| = \det ||P_{nm}|| \cdot \det ||x_i^{2m}|| = \det ||P_{nm}|| \cdot \prod_{i<k}^{N+1} (x_i^2 - x_k^2), \tag{21}$$

after which formula (20) takes the form

$$\mathrm{Tr}_N f(x^2) = \frac{\int_{-\infty}^{+\infty} \prod_{i=1}^{N+1} dx_i \left\{ \sum_{i=1}^{N+1} f(x_i^2) \right\} \prod_{i\neq k}^{N+1}(x_i^2 - x_k^2) \exp\left\{ -\hbar_N^{-1} \sum_{i=1}^{N+1}(ax_i^4 + bx_i^2) \right\}}{\int_{-\infty}^{+\infty} \prod_{i=1}^{N+1} dx_i \prod_{i\neq k}^{N+1}(x_i^2 - x_k^2) \exp\left\{ -\hbar_N^{-1} \sum_{i=1}^{N+1}(ax_i^4 + bx_i^2) \right\}}. \tag{22}$$

But from the random matrix theory we know that this expression can be rewritten as

$$\mathrm{Tr}_N f(x^2) = \frac{\int D[\Phi_N^\dagger, \Phi_N] \left\{ \mathrm{Tr} f(\Phi_N^\dagger \Phi_N) \right\} \exp\left\{ -\frac{1}{\hbar_N} \mathrm{Tr}\left[a(\Phi_N^\dagger \Phi_N)^2 + b\Phi_N^\dagger \Phi_N \right] \right\}}{\int D[\Phi_N^\dagger, \Phi_N] \exp\left\{ -\frac{1}{\hbar_N} \mathrm{Tr}\left[a(\Phi_N^\dagger \Phi_N)^2 + b\Phi_N^\dagger \Phi_N \right] \right\}}, \tag{23}$$

where the integration is performed over all $(N+1) \times (N+1)$ complex matrices Φ_N. It is well known [15] that the invariance of the subintegrals in eq.(23) under the transformation $\Phi_N \to U_1 \Phi_N U_2$ of the double unitary group $U(N+1) \times U(N+1)$ leads to the equality between eq.(22) and eq.(23).

Note also that the same steps hold if the monic classical polynomials P_n replace $\mathcal{P}_n$ in the above equations, then the equality of (20) with (23) is a special case of evaluation of connected correlation functions of $U(N+1)$ invariant operators [16].

Formula (23) establishes an exact correspondence between the random matrix model with quartic potential $V(\Phi_N^\dagger \Phi_N) = b\Phi_N^\dagger \Phi_N + a(\Phi_N^\dagger \Phi_N)^2$ (in which Φ_N is a $(N+1) \times (N+1)$ random complex matrix and $\Phi_N^\dagger$ is its adjoint) and quasi-exactly solvable anharmonic oscillator model with sextic potential $W_N(x^2) = [b^2 - (8 - 2\hbar_N)a]x^2 + 4abx^4 + 4a^2x^6$ (in which $N+1$ is the number of exactly calculable eigenvalues and corresponding eigenfunctions).

5 Rectangular random matrices

Let us now consider another class of quasi-exactly solvable models described by the hamiltonian

$$H_N = -\hbar_N^2 \frac{\partial^2}{\partial x^2} + \hbar_N^2 \frac{(4s-1)(4s-3)}{4x^2} + \{b^2 - [8 + (8s-4)\hbar_N]a\}x^2 + 4abx^4 + 4a^2x^6 \quad (24)$$

and defined on the positive half-axis $x \in [0, \infty]$. These models differ from the models (6) by the presence of an additional parameter s. If $s = 1/4$ then (24) reduce to (6). The exactly calculable wavefunctions in models (24) have the form

$$\Psi_n(x) = \mathcal{P}_n(x^2)(x^2)^{s-1/4} \exp\left\{-\frac{ax^4 + bx^2}{2\hbar_N}\right\}, \quad (25)$$

where $\mathcal{P}_n(x^2)$ are polynomials orthogonal with the weight $|x|^{4s-1} \exp[-(ax^4 + bx^2)/\hbar_N]$. Note however that the weight function contains now an additional factor $|x|^{4s-1}$, and therefore, instead of formula (22), we obtain

$$\mathrm{Tr}_N f(x^2) =$$
$$= \frac{\int_0^{+\infty} \prod_{i=1}^{N+1} x_i^{4s-1} dx_i \left\{\sum_{i=1}^{N+1} f(x_i^2)\right\} \prod_{i \neq k}^{N+1} (x_i^2 - x_k^2) \exp\left\{-\hbar_N^{-1} \sum_{i=1}^{N+1}(ax_i^4 + bx_i^2)\right\}}{\int_0^{+\infty} \prod_{i=1}^{N+1} x_i^{4s-1} dx_i \prod_{i \neq k}^{N+1} (x_i^2 - x_k^2) \exp\left\{-\hbar_N^{-1} \sum_{i=1}^{N+1}(ax_i^4 + bx_i^2)\right\}}. \quad (26)$$

There are two ways of transforming this expression into the fraction of matrix integrals. The first way is direct. We rewrite the product $\prod_{i=1}^{N+1} x_i^{4s-1}$ as $\exp\left\{(2s - 1/2) \sum_{i=1}^{N+1} \ln x_i^2\right\}$ and obtain the random matrix theory with a non-polynomial potential $V(\Phi_N^\dagger \Phi_N) = -(2s - 1/2)\ln(\Phi_N^\dagger \Phi_N) + \hbar_N^{-1}[b\Phi_N^\dagger \Phi_N + a(\Phi_N^\dagger \Phi_N)^2]$. The second way is more interesting. It is based on the observation that the expression (26) naturally arises in the model of random complex rectangular matrices Φ_N^s of the size $(N+1) \times 4s(N+1)$ with quartic potential $V(\bar{\Phi}_N^s \Phi_N^s) = \hbar_N^{-1}[b\bar{\Phi}_N^s \Phi_N^s + a(\bar{\Phi}_N^s \Phi_N^s)^2]$ (and without any singular term!).Here $\bar{\Phi}_N^s$ is the transpose of the conjugate of the matrix Φ_N^s. This enables one to assert that relation (23) holds for all quasi-exactly solvable models of the form (24) but the integration in the right-hand side is generally performed over all complex rectangular matrices of size $(N+1) \times 4s(N+1)$ [17, 18, 19, 20].

6 The large N limit

Up to now we considered N as an arbitrary non-negative integer. Now consider the case when $N \to \infty$. In large N limit the models (6) and (24) cease to be quasi-exactly solvable and become

exactly non-solvable. The N-dependence of their potentials disappears and they take the form

$$W(x^2) = (b^2 - 8a)x^2 + 4abx^4 + 4a^2x^6. \tag{27}$$

The quantized Planck constant $\hbar_N$ tends to zero and we obtain a typical semi-classical situation for all the spectrum of the model (27). In practice, however, we do not need the information of all the states because the left hand side of formula (23) requires the knowledge of only first $N + 1$ even excitations. For them we can use the standard semi-classical expressions. Substituting these expressions into formula (18) we obtain

$$\mathrm{Tr}_N f(x^2) = \sum_{n=0}^{N} \frac{\oint \frac{f(x^2)dx}{\sqrt{E_{2n} - W(x^2)}}}{\oint \frac{dx}{\sqrt{E_{2n} - W(x^2)}}}. \tag{28}$$

where E_{2n} can be found from the Bohr quantization rule

$$\frac{1}{2\pi} \oint dx \sqrt{E_{2n} - W(x^2)} = \hbar_N \left(2n + \frac{1}{2}\right). \tag{29}$$

Comparing (23) with (28) we arrive at the final relation

$$\frac{\int D[\Phi_N^\dagger, \Phi_N] \left\{ \mathrm{Tr} f(\Phi_N^\dagger \Phi_N) \right\} \exp\left\{ -\frac{1}{\hbar_N} \mathrm{Tr}\left[a(\Phi_N^\dagger \Phi_N)^2 + b\Phi_N^\dagger \Phi_N \right] \right\}}{\int D[\Phi_N^\dagger, \Phi_N] \exp\left\{ -\frac{1}{\hbar_N} \mathrm{Tr}\left[a(\Phi_N^\dagger \Phi_N)^2 + b\Phi_N^\dagger \Phi_N \right] \right\}} = \sum_{n=0}^{N} \frac{\oint \frac{f(x^2)dx}{\sqrt{E_{2n} - W(x^2)}}}{\oint \frac{dx}{\sqrt{E_{2n} - W(x^2)}}} \tag{30}$$

which can be considered as an alternative tool for doing calculations in the theory of large random matrices. It is worth stressing that eqs.(29), (30) correctly describe only the leading term of $1/N$ expansion for random matrix model under consideration. In order to obtain the non-leading terms, one should take into account the corrections to the main semi-classical approximations. It is interesting that, if the function $f(x^2)$ is a polynomial, than the computation of several terms in the $1/N$ expansion may be performed analytically. Indeed, for a polynomial $f(x^2)$, the main contribution to the right hand side of eq.(30) comes from large values of n, for which the analysis of the various elliptic integrals, typically appearing in semi-classical expansions, becomes trivial. Example of such computations will be provided in a separate paper.

7 The general case

It is known that, after an appropriate change of the initial variable x by a new variable $t = t(x)$, the hamiltonian

$$H_N = -\frac{\partial^2}{\partial x^2} + W_N(x) \tag{31}$$

of any one-dimensional quasi-exactly solvable model can be represented in the form

$$H_N = e^{-V_0(t)/2}\{C_{\alpha\beta}S_t^\alpha S_t^\beta + C_\alpha S_t^\alpha + C\}e^{V_0(t)/2}, \tag{32}$$

where S_t^α, $\alpha = -, 0, +$ are first order differential operators

$$S_t^- = \frac{\partial}{\partial t}, \qquad S_t^0 = t\frac{\partial}{\partial t} - \frac{N}{2}, \qquad S_t+ = t^2\frac{\partial}{\partial t} - Nt, \tag{33}$$

realizing a $(N+1)$-dimensional representation of algebra $sl(2)$ in the space of polynomials of order N. The basis of this representation is formed by monomials $\{1, t, \ldots, t^N\}$. Formula (31) means that, up to equivalence transformation, the hamiltonian of quasi-exactly solvable model is an element of the universal enveloping algebra of algebra $sl(2)$. For this reason, the linear span of functions $t^n e^{-V_0(t)/2}$, $n = 0, 1, \ldots, N$ is an invariant $(N+1)$-dimensional subspace $\mathcal{H}_N$ of Hilbert space and therefore the general solution of Schrödinger equation for H_N in $\mathcal{H}_N$ can be represented in the form $\Psi_n(x) = P_n(t(x))e^{-V_0(t(x))/2}$, $n = 0, 1, \ldots, N$, where $P_n(t)$ are certain polynomials of degree N. Because of the orthonormalizability of wavefunctions $\Psi_n(x)$, the polynomials $P_n(t)$ are orthogonal with the weight $e^{-V(t)} = (dx/dt)e^{-V_0(t)}$ which enables one to repeat the reasonings given above (see formulas (18) – (23)) and derive the generalized analog of formula (23)

$$\mathrm{Tr}_N f(x) = \frac{\int D\phi \, \{\mathrm{Tr} f(\phi)\} \exp[-\mathrm{Tr}\, V(\phi)]}{\int D\phi \exp[-\mathrm{Tr}\, V(\phi)]} \tag{34}$$

in which ϕ is assumed to be a random hermitian $(N+1) \times (N+1)$ matrix. Sometimes it is convenient to represent hermitean matrix ϕ in the form $\Phi^\dagger \Phi$ where Φ is an arbitrary complex matrix and $\Phi^\dagger$ is its adjoint. Then we arrive at direct generalizations of formula (23).

8   Random matrix theory, virial theorems and $sl(2)$ algebra

Assuming that N is finite consider function $f(x)$ of the form

$$f(x) = W_N(x) + \frac{1}{2}xW_N'(x). \tag{35}$$

Then, according to the well known virial theorem, the left hand side of formula (34) becomes

$$\mathrm{Tr}_N f(x) = \mathrm{Tr}_N H_N, \tag{36}$$

where H_N is the hamiltonian of the model (32). But from (31) it immediately follows that

$$\mathrm{Tr}_N f(x) = C_{\alpha\beta} g^{\alpha\beta} + (N+1)C, \tag{37}$$

where $g^{\alpha\beta}$ is the Killing – Cartan tensor of algebra $sl(2)$. Therefore,

$$\frac{\int D\phi \, \{\mathrm{Tr}[W_N(\phi) + \phi W_N'(\phi)]\} \exp[-\mathrm{Tr}\, V(\phi)]}{\int D\phi \exp[-\mathrm{Tr}\, V(\phi)]} = C^{\alpha\beta} g_{\alpha\beta} + (N+1)C. \tag{38}$$

Thus we have found a purely Lie algebraic expression for random matrix integrals associated with quasi-exactly solvable models.

9   Conclusion

We have completed the exposition of our approach to random matrix models associated with quasi-exactly solvable problems in quantum mechanics. We see that these models are actually simpler than general random matrix models and can be solved in a systematic way by constructing the semiclassical expansions for the associated quantum problems. Since the parameter of the semiclassical expansion — the quantized Planck constant is given by the formula $\hbar_N = 1/(N+1)$, this expansion is equivalent to the topological expansion (1/N expansion) in

random matrix models. This fact, which we consider as the main result of the present paper, provides a new way for doing calculations in quasi-exactly solvable random matrix models. It is worth stressing in this connection that these models are not some exotic 'monsters' but are rather simple and ordinary looking and are rather often discussed in the literature.

In conclusion, we would like to stress that in this paper we did not intend to present some systematic calculations of observables in quasi-exactly solvable random matrix models. Our aim was only to describe a general scheme, as to the applications, we leave them for the fortcoming publications (see e.g. ref. [21]).

10   Aknowledgements

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